The next two theorems are applications of Math 3220 analysis and the correspondence between $f: A \subseteq \mathbb{C} \to \mathbb{C}$, and $F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$: For each

$$f(x+iy) = u(x,y) + i\,v(x,y)$$
$$f: A \subseteq \mathbb{C} \to \mathbb{C},\ A \text{ open}$$

there corresponds

$$F(x,y) = (u(x,y),\ v(x,y))$$
$$F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2,\ A \text{ open}$$

<u>Theorem</u> (full CR Theorem) Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$, $\boldsymbol{z}_0 \in A$. Write

$$f(\boldsymbol{z}) = f(x + i\,y) = u(x,y) + i\,v(x,y)$$

where

$$u(x,y) = \mathrm{Re}(f(x+i\,y),\ v(x,y) = \mathrm{Im}(f(x+i\,y)$$

Then if f is complex differentiable at $\boldsymbol{z}_0 = x_0 + i\,y_0$ *if and only if* the following two conditions hold:

(1) The *Cauchy-Riemann equations* hold at (x_0, y_0):

$$u_x(x_0,y_0) = v_y(x_0,y_0)$$
$$u_y(x_0,y_0) = -v_x(x_0,y_0);$$

AND

(2) $F(x,y) = (u(x,y),\ v(x,y))$ is *Real differentiable* at (x_0, y_0) in the affine approximation sense you discussed in Math 3220. In particular real differentiability is implied by the condition that all of the partial derivatives u_x, y_y, v_x, v_y exist and are continuous in a neigborhood of (x_0, y_0).

HW 18abc

proof:

4200

- $f(z) = u(x,y) + i v(x,y)$
- $f'(z_0) = a + bi$ exists
- $\iff f(z_0+h) = f(z_0) + (a+bi)h + h\,\varepsilon(h)$ s.t. $\lim_{h\to 0} \varepsilon(h) = 0$

write $z = z_0 + h$,

$$\iff \lim_{z\to z_0} \frac{|f(z) - f(z_0) - (a+bi)(z-z_0)|}{|z-z_0|} = 0$$

$$\iff \lim_{(x+iy)\to(x_0+iy_0)} \frac{\left| u(x,y) + i v(x,y) - u(x_0,y_0) - i v(x_0,y_0) - (a+bi)\big((x-x_0) + i(y-y_0)\big) \right|}{|(x-x_0) + i(y-y_0)|} = 0$$

3220

$F(x,y) = (u(x,y), v(x,y))$

$$\iff \lim_{(x,y)\to(x_0,y_0)} \frac{\left\| \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} - \begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} - \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \right\|}{\left\| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\|} = 0$$

Math 3220

$\iff$ F is real-diff'ble @ (x_0,y_0) with differential matrix @ (x_0,y_0):

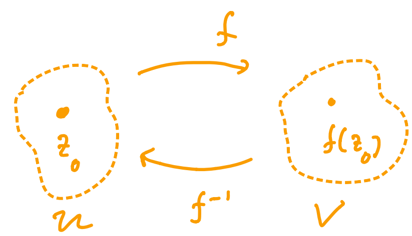
$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

∎

Theorem (Inverse function theorem) Let f be complex differentiable in a neighborhood of z_0, with $f'(z_0) \neq 0$ and $f'(z)$ continuous. Then there exist open sets U, V with $z_0 \in U$, $f(z_0) \in V$ such that $f: U \to V$ is a bijection and $f^{-1}: V \to U$ is also analytic. Furthermore

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}$$

$\forall z \in U$.



$f^{-1}(f(z)) = z$

proof: $\frac{d}{dz}$: $(f^{-1})'(f(z)) \cdot f'(z) = 1$

$\mathbb{C}$	$\mathbb{R}^2$
• $f: A \subset \mathbb{C} \to \mathbb{C}$	$F: A \subset \mathbb{R}^2 \to \mathbb{R}^2$

f analytic in a nbhd A of z_0, $f'(z)$ continuous on A, $f'(z_0) \neq 0$,

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$

$f'(z_0) = a + bi = f_x(z_0) = -i f_y(z_0)$

$(a = u_x(x_0,y_0) = v_y(x_0,y_0)$

$b = v_x(x_0,y_0) = -u_y(x_0,y_0))$

① $\Longrightarrow$

$F(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ real differentiable • in a nbhd of (x_0,y_0) & u_x, u_y, v_x, v_y cont

$DF(x,y) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ •

$DF(x_0,y_0) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is invertible • $(\det = a^2 + b^2 = |f'(z_0)|^2)$

② $\Downarrow$ 3220 inverse fcn thm

$\exists\, \mathcal{U}, V$ open, $(x_0,y_0) \in \mathcal{U} \subset A$; $F(x_0,y_0) \in V$ s.t. $F: \mathcal{U} \to V$ is a bijection, and $F^{-1}: V \to \mathcal{U}$ is real differentiable. Furthermore, by the 3220 chain rule and since

• $F^{-1}(F(x,y)) = (x,y) \quad \forall (x,y) \in \mathcal{U}$

the matrix product of derivative matrices

• $DF^{-1}(F(x,y))\, DF(x,y) = I$ ← identity matrix

In particular,

• $DF^{-1}(F(x,y)) = [DF(x,y)]^{-1}$

$\forall (x,y) \in \mathcal{U}$.

• Note, $DF(x,y) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, so

• $DF^{-1}(F(x,y)) = \frac{1}{\alpha^2+\beta^2} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

is also a rotation-dilation

③ $\Longleftarrow$

$\exists\, \mathcal{U}, V$ open, $z_0 \in A \subset \mathcal{U}$, $f(z_0) \in V$ s.t. $f: \mathcal{U} \to V$ is a bijection, and $f^{-1}: V \to \mathcal{U}$ has real differentiable counterpart F^{-1} whose partials satisfy CR eqtns

$\Rightarrow f^{-1}$ is analytic on V.

And since $f^{-1}(f(z)) = z$

$(f^{-1})'(f(z)) \cdot f'(z) = 1$

• $(f^{-1})'(f(z)) = \frac{1}{f'(z)}$, $\forall z \in \mathcal{U}$.

∎

Loose end: (applies to the hw problem 1.5.16)

<u>Theorem</u> Let A be an open connected set in $\mathbb{C}$, $f : A \to \mathbb{C}$ analytic, with $f'(\boldsymbol{z}) = 0 \;\forall\, \boldsymbol{z} \in A$. Then f is constant.

proof: For open sets, *connected* and *path-connected are equivalent.* Any continuous path connecting two points in A can be approximated with a continuously differentiable (C^1) path connecting the same two points. Let $\boldsymbol{z}_0$ be any fixed point in A. Let $\boldsymbol{z} \in A$ be any other point. Let γ be a C^1 curve,

$$\gamma : [a, b] \to A$$
$$\gamma(a) = \boldsymbol{z}_0$$
$$\gamma(b) = \boldsymbol{z}$$

Then by the fundamental theorem of Calculus (applied to the real and imaginary parts of f),

$$f(\boldsymbol{z}) - f(\boldsymbol{z}_0) = \int_a^b \frac{d}{dt} f(\gamma(t))\, \mathrm{d}t$$
$$= \int_a^b f'(\gamma(t))\, \gamma'(t)\, \mathrm{d}t$$
$$= \int_a^b 0\, \mathrm{d}t = 0.$$

QED

(Or, you showed in Math 3220 that a continuously differentiable function of several variables defined on an open connected set and with all partial derivatives equal to zero, is constant. That theorem applies here, since the partials of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ are zero if $f' \equiv 0$.)

Math 4200
Friday September 11

next HW 1.5 – start 1.6.

1.5: using CR equations to prove analyticity; harmonic functions and harmonic conjugates. We'll begin by finishing Wednesday's notes on the complete Cauchy-Riemann Theorem and the inverse function theorem.-- you've been using these already

Announcements:

Warm-up exercise Let $f(z) = e^z$
Find $f'(z) = e^z$?

first guess fn to check

$$\lim_{h\to 0} \frac{e^{z+h} - e^z}{h} = \lim_{h\to 0} \frac{e^z e^h - e^z}{h} = e^z \lim_{h_1+ih_2} \frac{e^{h_1+ih_2} - 1}{h_1+ih_2} \cdots$$

not so easy.

Clever way: Use CR.

$$f(x+iy) := e^x e^{iy} = \underbrace{e^x \cos y}_{u(x,y)} + i \underbrace{e^x \sin y}_{v(x,y)}$$

CR Thm $\Rightarrow$ If CR eqtns hold & if all partials are cont $\Rightarrow$ f is analytic. & $f'(z) = f_x = u_x + i v_x$

$u_x = v_y$? $\quad u_x = e^x \cos y, \; v_y = e^x \cos y$ ✓
$u_y = -v_x \quad u_y = -e^x \sin y \quad v_x = e^x \sin y$ ✓
& partials are cont.

$\Rightarrow e^z$ is analytic

$$(e^z)' = u_x + i v_x = e^x \cos y + i e^x \sin y = e^x e^{iy} = e^z!$$

end here!

(Sorry about my home internet failure, which cut our lecture in half, but we finished Wed notes. Monday's notes will essentially be today's, and HW3 will be due Friday instead of Wednesday. -Nick)